

ON CALCULATING THE MOTION OF THE INTERPHASE
BOUNDARY IN BODIES WITH SIMPLE SHAPES AND
VARIABLE PROPERTIES

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The nonlinear one-phase Stefan problem is considered under boundary conditions of the third kind and a constant ambient temperature.

The symmetrical problem of cooling with phase transformation in bodies of simple shapes (sphere or infinitely long cylinder with the radius l , or an infinitely large plate with the thickness $2l$), where the isothermal interphase boundary (T_0) is moving $0 \leq \xi(\tau) \leq l$ and the thermophysical properties are functions of the temperature as well as of the space coordinate (symmetrically), can be formulated as follows:

$$N(T, x) \frac{\partial T(x, \tau)}{\partial \tau} = \frac{\partial}{\partial x} \left[P(T, x) \frac{\partial T(x, \tau)}{\partial x} \right]; \quad (1)$$

$$0 \leq \tau \leq \tau_0; \quad 0 \leq x \leq \xi = \xi(\tau); \quad 0 \leq \xi \leq l;$$

$$T(x, 0) = T_0 = T(\xi, \tau) = \text{const}; \quad (2)$$

$$T(l, \tau) = T_0; \quad (3)$$

$$\alpha(T_s) \omega(0)(T_s - T_a) = P(T, 0) \frac{\partial T(0, \tau)}{\partial x}; \quad T_s = T(0, \tau); \quad (4)$$

$$T_a = \text{const} < T_0;$$

$$P(T_0, \xi) \frac{\partial T(\xi, \tau)}{\partial x} = q(\xi) \gamma(\xi) \omega(\xi) \frac{d\xi}{d\tau} \quad (5)$$

with $N(T, x) = c(T, x) \gamma(x) \omega(x)$; $P(T, x) = \lambda(T, x) \omega(x)$; and $\omega(x) = 1, 2\pi(l-x), 4\pi(l-x)^2$ respectively for a plate, a cylinder, or a sphere; with $x = 0$ at the body surface; τ_0 denoting the time after which a body has been completely frozen, when the interphase boundary reaches the center $\xi(\tau_0) = l$, with functions $c(T, x)$, $\gamma(x)$, $\lambda(T, x)$, $\alpha(T_s)$, $q(\xi)$, $\gamma(\xi)$ known, and with T_a denoting the ambient temperature.

The first integration from x to ξ at some instant of time τ , with condition (5) taken into account, yields

$$\frac{\partial T(x, \tau)}{\partial x} = -\frac{1}{P(T, x)} \left[\int_x^\xi N(T, y) \frac{\partial T}{\partial \tau} dy - q(\xi) \gamma(\xi) \omega(\xi) \frac{d\xi}{d\tau} \right] \quad (6)$$

or

$$\frac{\partial T(x, \tau)}{\partial x} = -\frac{1}{P(T, x)} \int_x^{\xi^+} N(T, y) \frac{\partial T(y, \tau)}{\partial \tau} dy, \quad (7)$$

where the plus sign indicates that the integral and the free term have been combined.

Integrating both sides of Eq. (7) with respect to the space coordinate from 0 to x at a fixed instant of time τ , with condition (4) taken into account, we obtain

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$$T(x, \tau) - T_a = -\frac{1}{\alpha(T_s)\omega(0)} \int_0^{\xi^+} N(T, y) \frac{\partial T(y, \tau)}{\partial \tau} dy - \int_0^x \frac{d\eta}{P(T, \eta)} \int_{\eta}^{\xi^+} N(T, y) \frac{\partial T(y, \tau)}{\partial \tau} dy. \quad (8)$$

At $x = \xi$ Eq. (8), with condition (2) taken into account, becomes

$$T_0 - T_a = -\frac{1}{\alpha(T_s)\omega(0)} \int_0^{\xi^+} N(T, y) \frac{\partial T}{\partial \tau} dy - \int_0^{\xi} \frac{d\eta}{P(T, \eta)} \int_{\eta}^{\xi^+} N(T, y) \frac{\partial T}{\partial \tau} dy \quad (9)$$

or

$$T_0 - T_a = - \int_{-0}^{\xi} \frac{d\eta}{P(T, \eta)} \int_{\eta}^{\xi^+} N(T, x) \frac{\partial T(x, \tau)}{\partial \tau} dx, \quad (10)$$

where the minus sign before 0 indicates that the integration interval extends beyond the body surface ($x = 0$) and, as a consequence, a term which accounts for the boundary condition at the body surface has been added to the integral.

Such a notation in Eq. (10) makes it possible to represent its right-hand side as a double integral over the region $[-0 \leq \eta \leq \xi; \eta \leq x \leq \xi^+]$, which in turn can be represented by another form of a double integral:

$$T_0 - T_a = - \int_0^{\xi^+} N(T, x) \frac{\partial T(x, \tau)}{\partial \tau} \left(\int_{-0}^x \frac{d\eta}{P(T, \eta)} \right) dx \quad (11)$$

or

$$T_0 - T_a = - \int_0^{\xi} N(T, x) \frac{\partial T(x, \tau)}{\partial \tau} \left[\frac{1}{\alpha(T_s)\omega(0)} + \int_0^x \frac{d\eta}{P(T, \eta)} \right] dx + q(\xi) \gamma(\xi) \omega(\xi) \left[\frac{1}{\alpha(T_s)\omega(0)} + \int_0^{\xi} \frac{dx}{P(T, x)} \right] \frac{d\xi}{d\tau}. \quad (12)$$

We integrate both sides of Eq. (12) with respect to time from $\tau = 0$ to $\tau = \tau^* < \tau_0$, we then reverse the sequence of integrations while replacing the integration variable τ by T in the first term and by ξ in the second term on the right-hand side of the equation, we then divide both sides by $(T_0 - T_a)$, and obtain a final formula for determining the time from the start of the freezing process to the instant when the interphase boundary ξ has moved to any position $0 < \xi(\tau^*) < l$:

$$\begin{aligned} \tau^* = & \frac{1}{T_0 - T_a} \int_0^{\xi^*} dx \int_{T(x, \tau^*)}^{T_0} N(T, x) \left[\frac{1}{\alpha(T, x)\omega(0)} + \int_0^x \frac{d\eta}{P(T, x, \eta)} \right] dT \\ & + \frac{1}{T_0 - T_a} \int_0^{\xi^*} q(\xi) \gamma(\xi) \omega(\xi) \left[\frac{1}{\alpha(\xi)\omega(0)} + \int_0^{\xi} \frac{dx}{P(\xi, x)} \right] d\xi, \end{aligned} \quad (13)$$

where $\xi^* \equiv \xi(\tau^*)$ and $T(x, \tau^*) \equiv F^*(x)$.

The length of time through which the interphase boundary moves can be determined according to formula (13), if an algebraic expression is given for the temperature distribution (profile) across the body thickness. Functions $\alpha(T, x)$, $P(T, x, \eta)$, $\alpha(\xi)$, $P(\xi, x)$ in formula (13) are found by substituting the temperature profiles for T in the temperature characteristics of these properties. When a fictitious linear temperature profile T across the body thickness is selected, for instance, then function $\alpha(T, x)$ is found by inserting into $\alpha(T_s)$ the expression

$$T_s = T_a + (T - T_a) \frac{\eta + k}{x + k} \equiv F_1(T, x),$$

function $P(T, x, \eta) \equiv \lambda(T, x, \eta)\omega(\eta)$ is found by substituting

$$t = T_a + (T - T_a) \frac{\eta + k}{x + k} \equiv F_2(T, x, \eta)$$

for T in $\lambda(T, x)$, function $\alpha(\xi)$ is found by inserting

$$T_s = T_a + (T_0 - T_a) \frac{k}{\xi + k} \equiv F_3(\xi)$$

into $\alpha(T_S)$, and function $P(\xi, x) \equiv \lambda(\xi, x)\omega(x)$ is found by substituting

$$t = T_a + (T_0 - T_a) \frac{x + k}{\xi + k} \equiv F_4(\xi, x)$$

for T in $\lambda(T, x)$; furthermore,

$$T(x, \tau^*) = T_a + (T_0 - T_a) \frac{x + k}{l + k} \equiv F^*(x),$$

where $k = \bar{\lambda}/\bar{\alpha}$ and $\bar{\lambda}, \bar{\alpha}$ are some fixed values of the thermal conductivity and of the heat transfer coefficient respectively.

In order to determine the time till a body is completely frozen, it is convenient to clear the values of N, P , and to write formula (13) in the form

$$\tau_0 = \frac{1}{T_0 - T_a} \int_0^l q(\xi) \gamma(\xi) (l - \xi)^{f-1} \left[\frac{1}{\alpha(\xi) l^{f-1}} + \int_0^{\xi} \frac{dx}{(l-x)^{f-1} \lambda(\xi, x)} \right] d\xi + \frac{1}{T_0 - T_a} \int_0^l dx \int_{F(x)}^{T_0} c(T, x) \gamma(x) (l-x)^{f-1} \left[\frac{1}{\alpha(T, x) l^{f-1}} + \int_0^x \frac{d\eta}{(l-\eta)^{f-1} \lambda(T, x, \eta)} \right]. \quad (14)$$

where $f = 1, 2$, or 3 respectively for a plate, a cylinder, or a sphere, with $F(x)$ denoting the final temperature profile at time $\tau = \tau_0$.

With a given linear temperature profile during freezing and with constant thermophysical properties ($c, \gamma_1, \gamma_2, \lambda, q, \alpha$), for instance, formula (14) becomes

$$\tau_0 = \frac{q\gamma_1 l}{(T_0 - T_c) f} \left(\frac{1}{\alpha} + \frac{l}{2\lambda} \right) + \frac{c\gamma_2 l^2}{(f+1) \left(l + \frac{\lambda}{\alpha} \right)} \left(\frac{1}{\alpha} + \frac{l}{3\lambda} \right). \quad (15)$$